A Note on the Effect of Zonal Boundaries on Equatorial Waves

T. D. DICKEY[1]

Geophysical Fluid Dynamics Laboratory, Princeton University, Princeton, New Jersey 08540

The purpose of this brief report is to determine the modification of equatorial waves of an ocean or atmosphere by the imposition of two infinite zonal walls. The region of interest lies within 30°S to 30°N. The geometry of two infinite zonal walls is relevant to numerical models in which walls are introduced along circles of latitude and possibly to the Caroline Basin (in the western Pacific). The numerical calculations agree with analytic solutions in the asymptotic limit.

Introduction

The analysis of equatorial waves utilizing linearized equations for inviscid, adiabatic, and hydrostatic flow on an unbounded beta plane was developed by *Matsuno* [1966] for the atmosphere and *Moore* [1968] for the ocean. This report concerns the effect of the location of two infinite zonal walls upon equatorial waves. The case of a single zonal wall, a case relevant to the Gulf of Guinea, has been studied by *Philander* [1977] and *Hickie* [1977]. The case of two zonal walls is relevant to numerical models in which walls (coasts) are introduced along circles of latitude [*Koss*, 1967; *Mak*, 1969] and possibly to the Caroline Basin (in the western Pacific). Here we consider a channel of varying width which is symmetric about the equator in the domain 30°S to 30°N.

Mathematical Formulation

The fluid is assumed to be inviscid, and the basic state is taken such that no mean currents are present. Further, the fluid is considered to conform with the Boussinesq approximation, to have a linear equation of state, and to be in the region of validity of the beta plane approximation. The equation for the meridional component of velocity as derived by *Matsuno* [1966], *Moore* [1968], and *Moore and Philander* [1976] can be written

$$v_{xx}' + v_{yy}' + (\sigma^2 - y^2)v' - \frac{iv_x'}{\sigma} = 0 \tag{1}$$

in nondimensional form. Eastward, northward, and upward coordinates are (x, y, z). A solution of the form $e^{-i\sigma t}$ has been assumed, where σ is frequency and t is time. The units of time and length are

$$[T] = (1/c\beta)^{1/2} \qquad [L] = (c/\beta)^{1/2} \tag{2}$$

where $\beta = 2\Omega/R$, Ω is the rotation rate of the earth, and R is the radius of the earth. Also, $c = (gh)^{1/2}$ is the phase speed, and h corresponds to the actual depth for the barotropic mode and to the equivalent depth for baroclinic modes. Equation (1) can be written in a self-adjoint form by introducing the integrating factor $e^{ix/2\sigma}$ [*Hickie*, 1977]. Let

$$v'(x, y) = e^{ix/2\sigma}v(x, y) \tag{3}$$

Then (1) is transformed into

$$v_{xx} + v_{yy} + (v/4\sigma^2) + (\sigma^2 - y^2)v = 0 \tag{4}$$

When separability of the form

$$v(x, y) = f(x)g(y) \tag{5}$$

is assumed, then

$$g_{yy} + (K - y^2)g = 0 \tag{6}$$

and

$$f_{xx} + [(1/4\sigma^2) + \sigma^2 - K]f = 0 \tag{7}$$

where the separation constant is K. Equation (6) is Weber's equation, and (7) can be rewritten as

$$f_{xx} + A_\nu^2 f = 0 \tag{8}$$

where

$$A_\nu^2 = [(1/2\sigma) - \sigma]^2 - 2\nu \tag{9}$$

and

$$K = 2\nu + 1$$

The solution of this equation is

$$f_\nu(x) = \exp(\pm iA_\nu x) \tag{10}$$

To solve (6), a change of variables is made, $y = Y/(2)^{1/2}$, so that

$$g_{YY} + (\nu + \tfrac{1}{2} - Y^2/4)g = 0 \tag{11}$$

The boundary conditions for the problem are

$$g(Y_0) = 0 \tag{12}$$

at lower wall $v = 0$, and

$$g(Y_1) = 0 \tag{13}$$

at upper wall $v = 0$. The general form of the solution of (4) is

$$v(x, y) = \exp(ik_m^{\pm}x)g_{\nu_m}(y) \tag{14}$$

where

$$k_m^{\pm} = \frac{1}{2\sigma} \pm \left[\left(\frac{1}{2\sigma} - \sigma\right)^2 - 2\nu_m\right]^{1/2} \tag{15}$$

is the dispersion relation. The ν_m are determined by using (11) and boundary conditions (12) and (13).

The general solution of (11) is a parabolic cylinder function. For the particular computations undertaken in this work it was found that the solution in the form of the confluent hypergeometric function was the most convenient from a computer programing aspect. Letting $a = -\nu/2 - \frac{1}{4}$, (11) becomes

$$g_{YY} - (Y^2/4 + a)g = 0 \tag{16}$$

with the solution

[1] Now at Rosenstiel School of Marine and Atmospheric Science, Division of Meteorology and Physical Oceanography, University of Miami, Miami, Florida 33149.

Copyright © 1978 by the American Geophysical Union.

Paper number 8C0179.
0148-0227/78/078C-0179$01.00

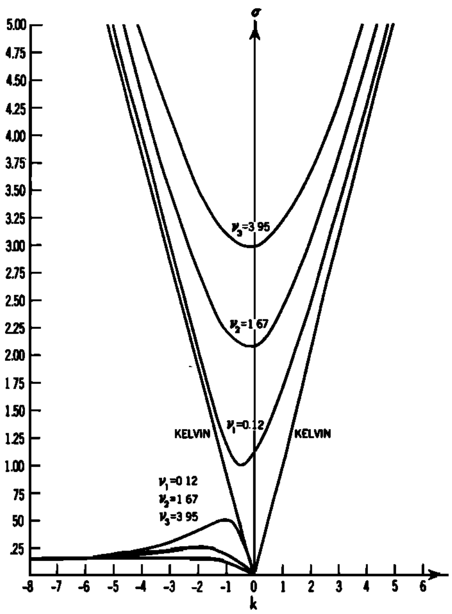

Fig. 1. Dispersion diagrams for the first baroclinic mode of a basin with coasts at 5°S and 5°N and $\nu_1 = 0.12$, $\nu_2 = 1.67$, and $\nu_3 = 3.95$.

$$g(Y) = c_1 g_1(Y) + c_2 g_2(Y) \tag{17}$$

or

$$g(a, Y) = c_1 e^{-Y^2/4} M(a/2 + \tfrac{1}{4}, \tfrac{1}{2}, Y^2/2) + c_2 Y e^{-Y^2/4} M(a/2 + \tfrac{3}{4}, \tfrac{3}{2}, Y^2/2) \tag{18}$$

where the general form of M, the Kummer function, is

$$M(\alpha, \beta, \xi) = 1 + \sum_{i=1}^{N} \frac{(\alpha)_i \xi^i}{(\beta)_i i!} \tag{19}$$

where

$$(\alpha)_i = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + i - 1)$$

TABLE 1. Equivalent Depths and Nondimensional Northern Coastline Latitudes for the First Two Baroclinic Modes

Mode	h_n, cm	Y_n (5°N)
1	60	1.7*
2	20	2.2

*Nondimensional.

and

$$(\beta)_i = \beta(\beta + 1)(\beta + 2) \cdots (\beta + i - 1)$$

The first term of (18) is an even function, while the second is odd. When the boundary conditions of the problem are used, the form of the constants appropriate for these conditions is

$$c_1 = g_2(a, Y_0) \qquad c_2 = -g_1(a, Y_0) \tag{20}$$

Thus it follows that

$$g(a, Y_0) = 0 \tag{21}$$

and

$$g(a, Y_1) = g_2(a, Y_0)g_1(a, Y_1) - g_1(a, Y_0)g_2(a, Y_1) = 0 \tag{22}$$

Equation (22) is used to determine the eigenvalues.

DISPERSION RELATION

The case of infinite zonal walls located symmetrically about the equator is considered. The primary motivation of this calculation is to determine the sensitivity of the dispersion relation and the meridional velocity structure to the location of walls. However, the first and second baroclinic modes of a region with walls located at 5°S and 5°N are also considered.

The indicial equation (22) is used, and we define the northern boundary as Y_0 and the southern boundary as $Y_1 = -Y_0$. The fact that $g_1(a, Y_1) = g_1(a, Y_0)$ by virtue of the evenness of g_1 and that $g_2(a, Y_1) = -g_2(a, Y_0)$ by the oddness of g_2 results

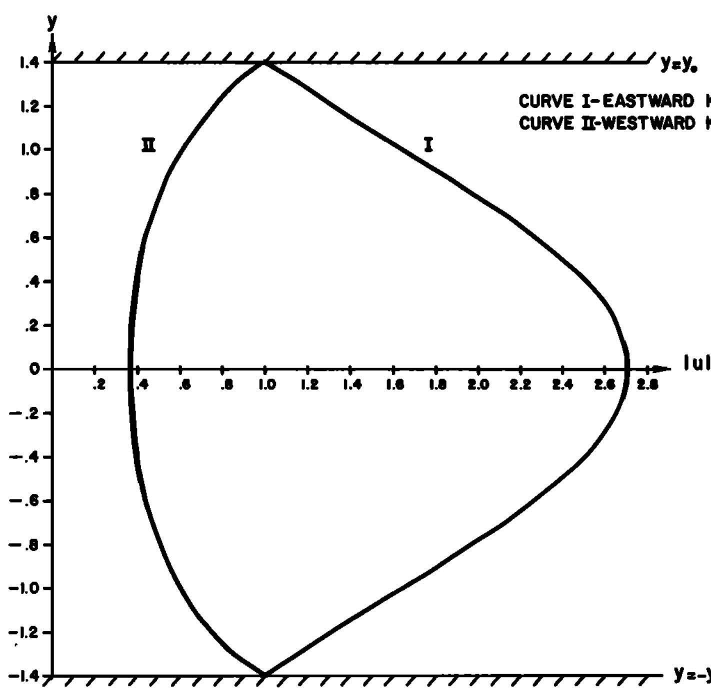

Fig. 2. Meridional structure of the zonal velocity for an eastward propagating Kelvin wave (curve I) and for a westward propagating Kelvin wave (curve II) with walls at $y_0 = \pm 1.4$.

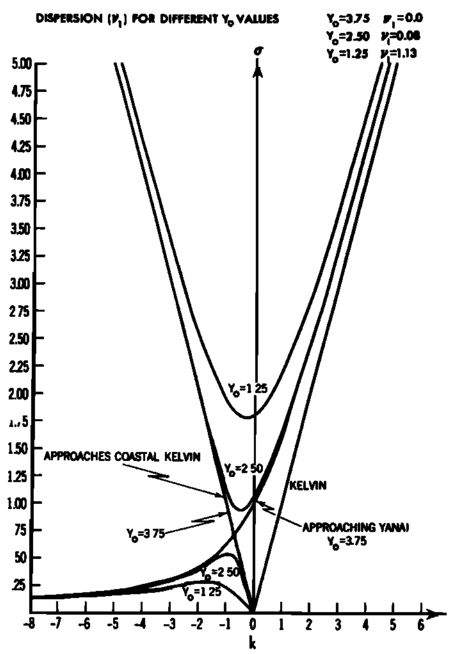

Fig. 3. Dispersion diagrams for the first mode ν_1 using three locations of the coasts $Y_0 = \pm 1.25$, $Y_0 = \pm 2.50$, and $Y_0 = \pm 3.75$.

in the indicial equation

$$g_2(-\nu - \tfrac{1}{2}, Y_0)g_1(-\nu - \tfrac{1}{2}, Y_0) = 0 \tag{23}$$

It follows that

$$M(-\nu/2 + \tfrac{1}{2}, \tfrac{3}{2}, Y_0^2/2)M(-\nu/2, \tfrac{1}{2}, Y_0^2/2) = 0 \tag{24}$$

A computer program was developed to determine those ν_m which satisfy (24) for particular values of Y_0. As can be seen from (19), the Kummer function $M(\alpha, \beta, \xi)$ is a power series which is convergent for $-\infty < \xi < \infty$. The rate of convergence is of concern, since the values of ν_m would be affected by insufficient accuracy in the determination of $M(\alpha, \beta, \xi)$. Monitoring the ratio of the value of the $n + 1$ term to the nth term and the change in the value of the summation as the upper limit of the summation was increased, it was found that

TABLE 2. Meridional Wave Numbers ν_1, ν_2, ν_3, and ν_4 for Various Locations of the Southern and Northern Coasts

Mode	Y_0	ν_1	ν_2	ν_3	ν_4
	1.25	1.13	5.93	13.83	24.89
1	2.35	0.12	1.67	3.95	7.09
	2.50	0.08	1.50	3.54	6.32
2	3.10	0.02	1.15	2.56	4.39
	3.75	0	1.03	2.15	3.46
	5.00	0	1.00	2.00	3.02
	6.25	0	1.00	2.00	3.01
	7.50	0	1.00	2.00	3.00
	10.00	0	1.00	2.00	3.00

$N = 20$ was sufficient for the domain of Y considered. The equivalent depths for the first two baroclinic modes as calculated by Katz [see *Moore and Philander,* 1976] using stratification in the equatorial Atlantic are shown in Table 1. The meridional wave numbers were then determined for several locations of the walls along with the ν_m appropriate for the first and second baroclinic modes with walls at 5°S and 5°N. The dispersion diagram for the first baroclinic mode is shown in Figure 1. The upper three curves represent inertia-gravity waves, while the lower three curves indicate Rossby waves. Kelvin waves are found where $\sigma = \pm k$ [e.g., *Matsuno,* 1966]. The meridional structure of the zonal velocity for $\sigma = k$ (eastward propagation) is Gaussian with a maximum at the equator (Figure 2) or

$$u = Ce^{-Y^2/2} \tag{25}$$

For $\sigma = -k$ (westward propagation), maxima occur at the walls as

$$u = -Ce^{Y^2/2} \tag{26}$$

The former case is permissible for an infinite plane; however, the latter occurs only under conditions where zonal walls are present.

The effect of moving coasts to higher latitudes upon the meridional wave numbers is indicated in Table 2. By increasing Y_0 or looking at higher baroclinic modes, the meridional wave numbers approach 0, 1, 2, 3, $\cdots$. This conforms with the

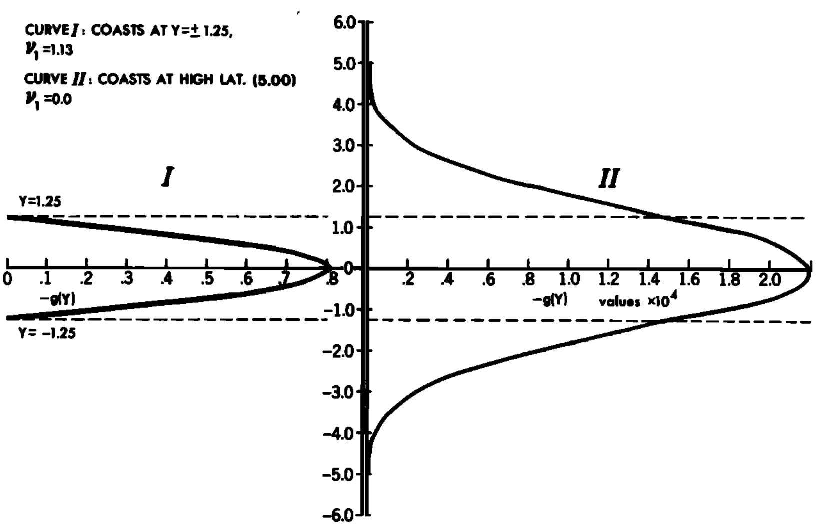

Fig. 4. Meridional velocity structure function $g(Y)$ versus Y. Curve I represents the case where the coasts are at 5°S and 5°N. The first baroclinic mode and $\nu_1 = 1.13$ are used. Curve II represents the case where the coasts are at higher latitudes ($Y_0 = \pm 5.00$). The first mode $\nu_1 = 0.0$ is used.

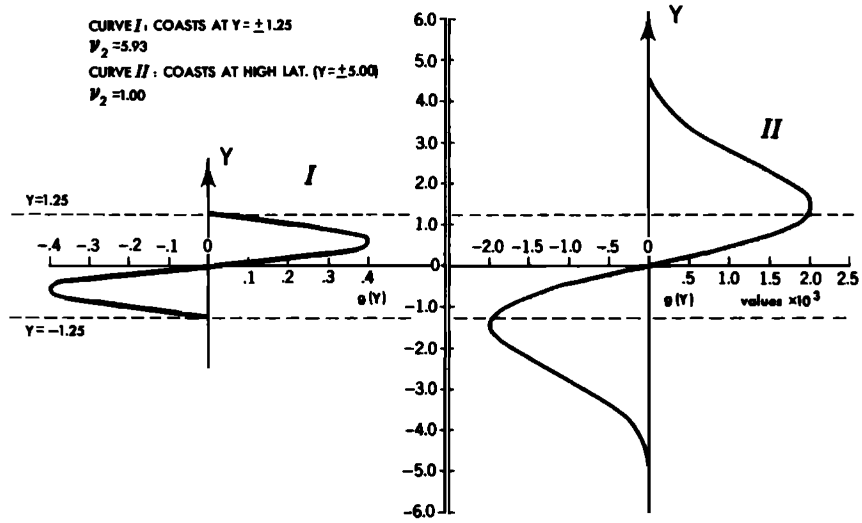

Fig. 5. Meridional velocity structure function $g(Y)$ versus Y. Curve I represents the case where the coasts are at 5°S and 5°N. The first baroclinic mode and $\nu_2 = 5.93$ are used. Curve II represents the case where the coasts are at higher latitudes ($Y_0 = \pm 5.00$). The second mode $\nu_2 = 1.00$ is used.

results of the analytic solution for the unbounded basin. Figure 3 demonstrates the effect of the location of the walls upon the dispersion diagrams for the first meridional mode. Increasing Y_0 results in lower frequencies for the inertia-gravity waves while the frequencies of the Rossby waves for a given wave number are being increased. An interesting feature is indicated by the present analysis. The Yanai wave and a coastally trapped Kelvin wave result in the limit as Y_0 becomes large ($\nu_1 \rightarrow 0$) as seen in Figure 3. The solution of (11) for the present case is given by

$$g_\nu(Y) = Y_0 e^{-Y_0^2/4} M(-\nu/2 + \tfrac{1}{2}, \tfrac{3}{2}, Y_0^2/2) e^{-Y^2/4} M(-\nu/2, \tfrac{1}{2}, Y^2/2)$$
$$- e^{-Y_0^2/4} M(-\nu/2, \tfrac{1}{2}, Y_0^2/2) Y e^{-Y^2/4} M(-\nu/2 + \tfrac{1}{2}, \tfrac{3}{2}, Y^2/2) \quad (27)$$

Both symmetric and antisymmetric modes are included within this solution. The effect of moving the coasts from $Y = \pm 1.25$ to a high latitude may be seen in the plots of $g(Y)$ versus Y for the first two meridional modes (Figures 4 and 5).

Analogous computations were done for the case of a wall fixed at the equator and a second wall located at various latitudes. This configuration (with the northern wall at 5°N) may have relevance for a region such as the Caroline Basin (western Pacific). Results indicate the elimination of the symmetric modes. Also, the Yanai wave is eliminated so that the coastal Kelvin wave on the dispersion diagram is not deformed.

The solution of the infinite half-plane problem [*Philander*, 1977; *Hickie*, 1977] agrees with the present solution when one wall is fixed at the equator and the other is moved to a high latitude.

Acknowledgments. I would like to thank S. G. H. Philander for suggesting this study and for his comments concerning the manuscript. Several discussions with B. P. B. Hickie were most helpful. Thanks are also due J. E. Geisler for reading the manuscript and for his suggestions. The manuscript was typed by Rosalee Sierra and Jere Green. Support was provided by National Science Foundation grant ATM 75-19326 and by the Rosenstiel Research Fund.

REFERENCES

Hickie, B. P. B., The effects of coastal geometry on equatorial waves (free modes of the Gulf of Guinea), submitted to *J. Fluid Mech.*, 1977.

Koss, W. J., Further theoretical considerations of tropospheric wave motions in equatorial latitudes, *Mon. Weather Rev.*, *95*(5), 283–297, 1967.

Mak, M. K., Laterally driven stochastic motions in the tropics, *J. Atmos. Sci.*, *26*, 41–64, 1969.

Matsuno, T., Quasi-geostrophic motions in the equatorial area, *J. Meteorol. Soc. Jap.*, *44*, 25–42, 1966.

Moore, D. W., Planetary-gravity waves in an equatorial ocean, Ph.D. thesis, 201 pp., Harvard Univ., Cambridge, Mass., 1968.

Moore, D. W., and S. G. H. Philander, *The Sea*, chap. 8, pp. 319–361, John Wiley, New York, 1976.

Philander, S. G. H., The effects of coastal geometry on equatorial waves (forced waves in the Gulf of Guinea), *J. Mar. Res.*, *35*(3), 509–523, 1977.

(Received January 23, 1978;
accepted January 30, 1978.)